Classical Algorithms for Reasoning and Explanation in Description Logics

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German Physicists

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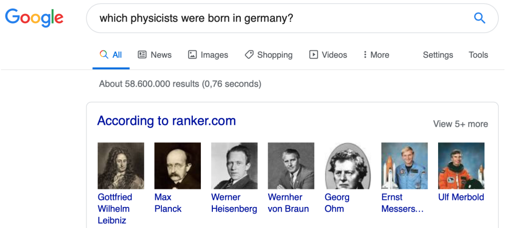

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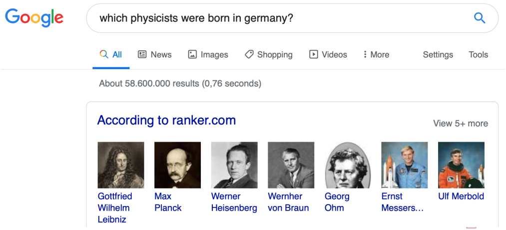

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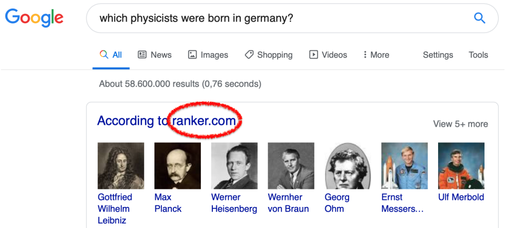

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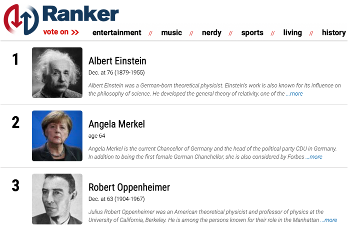

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Science and technology in Germany - Wikipedia

https://en.wikipedia.org › wiki › Science_and_technology_in_Germany ▾

German science **have been** very significant and research and development efforts form an ... They **were** preceded by such key **physicists** as Hermann von Helmholtz, Joseph von Fraunhofer, and Gabriel Daniel Fahrenheit, among others . Wilhelm ... Wladimir Köppen (1846–1940) **was** an eclectic Russian-**born** botanist and ...

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- In some applications wrong/missed results cannot be tolerated
  - medicine, banking, autonomous driving,. . .

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  - like in mathematics, it is well-defined what each formula means
- Main advantage: an answer can be obtained by combinding several sources of information (formulas)

Description Logics

- Example:

F1 = "Albert Einstein was a physicist"
F2 = "Albert Einstein was born in Ulm"
F3 = "Ulm is a city in Germany"

⇒ "Albert Einstein was a German Physicists"

DLs @ RW

- Reasoning Web summer school hosted many courses on DLs:
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  - explantion: axiom-pinpointing methods
- Main focus: correctness, complexity, optimizations

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Vocabulary of $\mathcal{ALC}$

- The vocabulary of DL $\mathcal{ALC}$ consists of:
  - Concept names (atomic concepts): A, B,...
  - Role names (atomic roles): R, S, H,...
  - Individual names (individuals): a, b, c,...
  - Logical symbols: $\top$, $\bot$, $\neg$, $\sqcap$, $\sqcup$, $\forall$, $\exists$.

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- Roles represent relations between things:
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- Individuals represent concrete (unique) objects:
  - *germany* – the country of Germany
  - *john* – the person John

Complex Concepts of $\mathcal{ALC}$

- Complex concepts are built using concept constructors:
  - $\top$ (top) is a concept that represents all objects in the world
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  - $C \sqcap D$ (conjunction) are the common objects of C and D
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  - $Male \sqcap \forall hasChild.\neg Male$ – all male beings all of whose children are not male

Axioms of $\mathcal{ALC}$

- Description Logic axioms postulate facts about concepts, roles, or individuals:
  - $C \sqsubseteq D$ (concept inclusion) states that every member of C is also a member of D
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  - $bornIn(einstein, ulm)$ – Albert Einstein was born in Ulm

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- Then its TBox $= \{(ax1), (ax2)\}$, its ABox $= \{(ax3)\}$

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- Example: define $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ as follows:
  - $\Delta^{\mathcal{I}} = \{a, b\}$
  - $Human^{\mathcal{I}} = \{a, b\}$, $Male^{\mathcal{I}} = \{a\}$, $Female^{\mathcal{I}} = \{b\}$
  - $hasChild^{\mathcal{I}} = \{\langle a, b\rangle\}$
  - $john^{\mathcal{I}} = a$, $mary^{\mathcal{I}} = b$

Interpretation of Complex Concepts

- The interpretation function $\cdot^{\mathcal{I}}$ can be recursively extended to complex concepts as follows:
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  - $\top^{\mathcal{I}} =$

Interpretation of Complex Concepts

- The interpretation function $\cdot^{\mathcal{I}}$ can be recursively extended to complex concepts as follows:
  - $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$
  - $\bot^{\mathcal{I}} = \emptyset$
  - $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$
  - $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$
  - $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
  - $(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y : (x, y) \in R^{\mathcal{I}} \ \&\ y \in C^{\mathcal{I}}\}$
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- Example: let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{a, b\}$, $Male^{\mathcal{I}} = \{a\}$, $Female^{\mathcal{I}} = \{b\}$, $hasChild^{\mathcal{I}} = \{\langle a, b\rangle\}$. Then:
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- Example: let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{a, b\}$, $\mathit{Male}^{\mathcal{I}} = \{a\}$, $\mathit{Female}^{\mathcal{I}} = \{b\}$, $\mathit{hasChild}^{\mathcal{I}} = \{\langle a, b\rangle\}$. Then:
  - $\top^{\mathcal{I}} = \{a, b\}$
  - $(\mathit{Male} \sqcap \mathit{Female})^{\mathcal{I}} =$

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  - $\top^\mathcal{I} = \{a, b\}$
  - $(Male \sqcap Female)^\mathcal{I} = \emptyset$

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  - $\top^{\mathcal{I}} = \{a, b\}$
  - $(Male \sqcap Female)^{\mathcal{I}} = \emptyset$
  - $(Male \sqcup Female)^{\mathcal{I}} =$

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  - $\top^{\mathcal{I}} = \{a, b\}$
  - $(Male \sqcap Female)^{\mathcal{I}} = \emptyset$
  - $(Male \sqcup Female)^{\mathcal{I}} = \{a, b\}$

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- Example: let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{a, b\}$, $Male^{\mathcal{I}} = \{a\}$, $Female^{\mathcal{I}} = \{b\}$, $hasChild^{\mathcal{I}} = \{\langle a, b\rangle\}$. Then:
  - $\top^{\mathcal{I}} = \{a, b\}$
  - $(Male \sqcap Female)^{\mathcal{I}} = \emptyset$
  - $(Male \sqcup Female)^{\mathcal{I}} = \{a, b\}$
  - $(\neg Male)^{\mathcal{I}} =$

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  - $\top^{\mathcal{I}} = \{a, b\}$
  - $(\mathit{Male} \sqcap \mathit{Female})^{\mathcal{I}} = \emptyset$
  - $(\mathit{Male} \sqcup \mathit{Female})^{\mathcal{I}} = \{a, b\}$
  - $(\neg \mathit{Male})^{\mathcal{I}} = \{b\}$

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  - $\top^{\mathcal{I}} = \{a, b\}$
  - $(Male \sqcap Female)^{\mathcal{I}} = \emptyset$
  - $(Male \sqcup Female)^{\mathcal{I}} = \{a, b\}$
  - $(\neg Male)^{\mathcal{I}} = \{b\}$
  - $(Male \sqcap \neg Female)^{\mathcal{I}} =$

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  - $\top^{\mathcal{I}} = \{a, b\}$
  - $(Male \sqcap Female)^{\mathcal{I}} = \emptyset$
  - $(Male \sqcup Female)^{\mathcal{I}} = \{a, b\}$
  - $(\neg Male)^{\mathcal{I}} = \{b\}$
  - $(Male \sqcap \neg Female)^{\mathcal{I}} = \{a\}$

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  - $(\exists hasChild.Female)^{\mathcal{I}} =$

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  - $(\exists hasChild.Female)^{\mathcal{I}} = \{a\}$

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  - $(\exists hasChild.Female)^\mathcal{I} = \{a\}$
  - $(\forall hasChild.Female)^\mathcal{I} =$

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- Example: let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ with $\Delta^\mathcal{I} = \{a, b\}$, $Male^\mathcal{I} = \{a\}$, $Female^\mathcal{I} = \{b\}$, $hasChild^\mathcal{I} = \{\langle a, b\rangle\}$. Then:
  - $(\exists hasChild.Female)^\mathcal{I} = \{a\}$
  - $(\forall hasChild.Female)^\mathcal{I} = \{a, b\}$ (!!!)

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  - $(\exists hasChild.Female)^\mathcal{I} = \{a\}$
  - $(\forall hasChild.Female)^\mathcal{I} = \{a, b\}$ (!!!)
  - $(\forall hasChild.Male)^\mathcal{I} =$

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  - $(\exists hasChild.Female)^{\mathcal{I}} = \{a\}$
  - $(\forall hasChild.Female)^{\mathcal{I}} = \{a, b\}$ (!!!)
  - $(\forall hasChild.Male)^{\mathcal{I}} = \{b\}$ (!!!)

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  - $(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y : (x, y) \in R^{\mathcal{I}} \ \& \ y \in C^{\mathcal{I}}\}$
  - $(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y.\ (x, y) \in R^{\mathcal{I}} \ \Rightarrow \ y \in C^{\mathcal{I}}\}$
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  - $(\exists hasChild.Female)^{\mathcal{I}} = \{a\}$
  - $(\forall hasChild.Female)^{\mathcal{I}} = \{a, b\}$ (!!!)
  - $(\forall hasChild.Male)^{\mathcal{I}} = \{b\}$ (!!!)
  - $(\exists hasChild.\forall hasChild.\bot)^{\mathcal{I}} =$

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  - $(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I}$
  - $(\exists R.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \exists y : (x, y) \in R^\mathcal{I} \ \&\ y \in C^\mathcal{I}\}$
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- Example: let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ with $\Delta^\mathcal{I} = \{a, b\}$, $Male^\mathcal{I} = \{a\}$, $Female^\mathcal{I} = \{b\}$, $hasChild^\mathcal{I} = \{\langle a, b\rangle\}$. Then:
  - $(\exists hasChild.Female)^\mathcal{I} = \{a\}$
  - $(\forall hasChild.Female)^\mathcal{I} = \{a, b\}$ (!!!)
  - $(\forall hasChild.Male)^\mathcal{I} = \{b\}$ (!!!)
  - $(\exists hasChild.\forall hasChild.\bot)^\mathcal{I} = \{a\}$

Interpretation of Axioms

- An interpretation can either satisfy an axiom ($\mathcal{I} \models \alpha$) or violate it ($\mathcal{I} \not\models \alpha$):
  - $\mathcal{I} \models C \sqsubseteq D \quad \text{iff} \quad C^\mathcal{I} \subseteq D^\mathcal{I}$
  - $\mathcal{I} \models C \equiv D \quad \text{iff} \quad C^\mathcal{I} = D^\mathcal{I}$
  - $\mathcal{I} \models C(a) \quad \text{iff} \quad a^\mathcal{I} \in C^\mathcal{I}$
  - $\mathcal{I} \models R(a, b) \quad \text{iff} \quad (a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$

Interpretation of Axioms

- An interpretation can either satisfy an axiom ($\mathcal{I} \models \alpha$) or violate it ($\mathcal{I} \not\models \alpha$):
  - $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
  - $\mathcal{I} \models C \equiv D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$
  - $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$
  - $\mathcal{I} \models R(a, b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
- Example: let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{a, b\}$, $\mathit{Male}^{\mathcal{I}} = \{a\}$, $\mathit{Female}^{\mathcal{I}} = \{b\}$, and $\mathit{hasChild}^{\mathcal{I}} = \{\langle a, b\rangle\}$, $\mathit{john}^{\mathcal{I}} = a$, $\mathit{mary}^{\mathcal{I}} = b$. Then:

Interpretation of Axioms

- An interpretation can either satisfy an axiom ($\mathcal{I} \models \alpha$) or violate it ($\mathcal{I} \not\models \alpha$):
  - $\mathcal{I} \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
  - $\mathcal{I} \models C \equiv D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$
  - $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$
  - $\mathcal{I} \models R(a, b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
- Example: let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta^{\mathcal{I}} = \{a, b\}$, $Male^{\mathcal{I}} = \{a\}$, $Female^{\mathcal{I}} = \{b\}$, and $hasChild^{\mathcal{I}} = \{\langle a, b\rangle\}$, $john^{\mathcal{I}} = a$, $mary^{\mathcal{I}} = b$. Then:
  - $\mathcal{I}$ $\quad Male \sqsubseteq Female$

Interpretation of Axioms

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  - $\mathcal{I} \models C \equiv D$ iff $C^\mathcal{I} = D^\mathcal{I}$
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  Indeed:
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- Proving unsatisfiability is harder: one has to prove that $\mathcal{I} \not\models \mathcal{O}$ for every interpretation $\mathcal{I}$!

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- An ontology $\mathcal{O}$ entails an axiom α ($\mathcal{O} \models \alpha$) if for every $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{O}$ we have $\mathcal{I} \models \alpha$.
- Example: $\mathcal{O} = \{A \sqsubseteq B,\ B \sqsubseteq C\} \models A \sqsubseteq C$
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Outline

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- We should aim at detecting as much errors as possible automatically

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     2. $\mathit{GrandParent} \equiv \exists \mathit{hasChild}.\mathit{Parent}$
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- Concept subsumption: $\mathcal{O} \models \mathit{Parent} \sqsubseteq \mathit{GrandParent}$? No:
  - $\mathcal{I} \not\models \mathit{Parent} \sqsubseteq \mathit{GrandParent}$.

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  - Hence, $\exists y : \langle x, y \rangle \in hashChild^{\mathcal{I}}$ & $y \in Parent^{\mathcal{I}}$.

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  - Since $x \in \mathit{GrandParent}^{\mathcal{I}}$ was arbitrary, we proved that $\mathit{GrandParent}^{\mathcal{I}} \subseteq \mathit{Parent}^{\mathcal{I}}$.

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  - Since $x \in GrandParent^{\mathcal{I}}$ was arbitrary, we proved that $GrandParent^{\mathcal{I}} \subseteq Parent^{\mathcal{I}}$.
  - Since $\mathcal{I} \models \mathcal{O}$ was arbitrary, we proved that $\mathcal{O} \models GrandParent \sqsubseteq Parent$.

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  - $\mathcal{O} \not\models \mathit{Parent}(\mathit{mary})$ – $\mathit{mary}^{\mathcal{I}} \notin \mathit{Parent}^{\mathcal{I}}$
- Instance checking: what are the instances of $\neg\mathit{Parent}$?
  - $\mathcal{O} \not\models (\neg\mathit{Parent})(\mathit{john})$ – Exercise
  - $\mathcal{O} \not\models (\neg\mathit{Parent})(\mathit{mary})$ – Exercise

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- Use a reduction!
  - if we solve one problem, we solve all of them!

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Definition (Reduction)

$P_1 : X \to \{yes, no\}$ is reducible to $P_2 : Y \to \{yes, no\}$ if $\exists$ an algorithm $R : X \to Y$ (a reduction) such that $\forall x \in X$:

- if $P_1(x) = yes$ then $P_2(R(x)) = yes$
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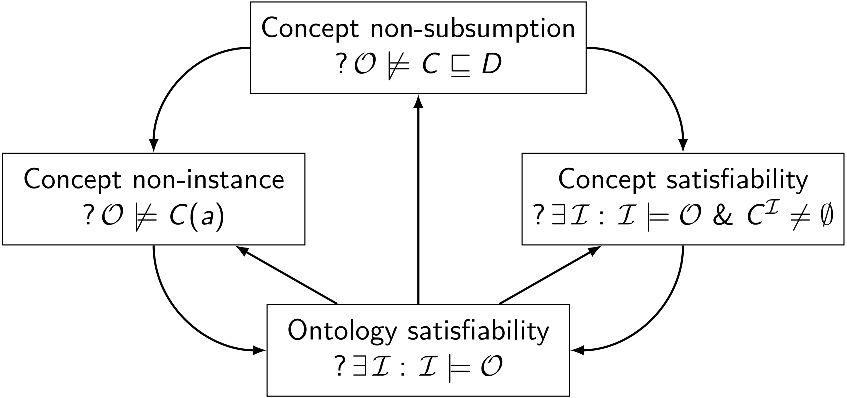
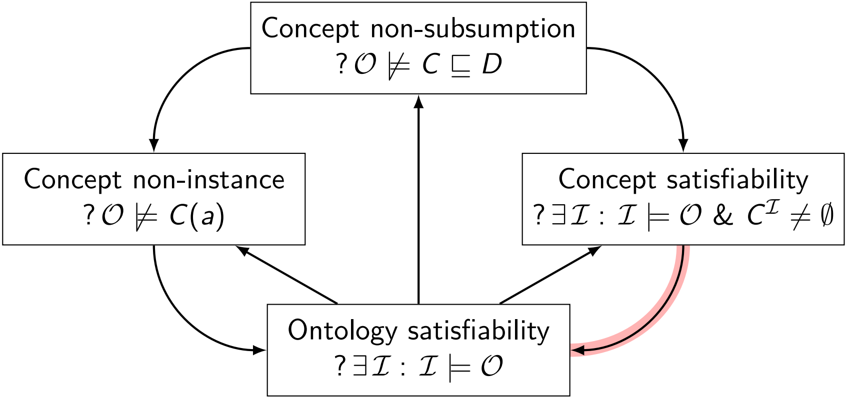
If R is polynomial then P_1 is polynomially reducible to P_2.

- polynomial reductions are of a most interest

Reduction between Reasoning Problems



Concept Satisfiability $\Rightarrow$ Ontology Satisfiability



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C is satisfiable w.r.t. $\mathcal{O}$ iff $\mathcal{O} \cup \{C(a)\}$ is satisfiable for every a not appearing in $\mathcal{O}$.

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- $(\Rightarrow)$: if C is satisfiable w.r.t. $\mathcal{O}$ then there exists $\mathcal{I} \models \mathcal{O}$ such that $C^{\mathcal{I}} \neq \emptyset$. That is, there exists some $x \in C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$.

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  - $a^{\mathcal{J}} = x \in C^{\mathcal{I}} \in \Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$

  Then $\mathcal{J} \models \mathcal{O}$ and $\mathcal{J} \models C(a)$.

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Proof.

- $(\Leftarrow)$: If $\mathcal{O} \cup \{C(a)\}$ is satisfiable then there exists a model $\mathcal{I} \models \mathcal{O} \cup \{C(a)\}$.

Concept Satisfiability $\Rightarrow$ Ontology Satisfiability

Lemma

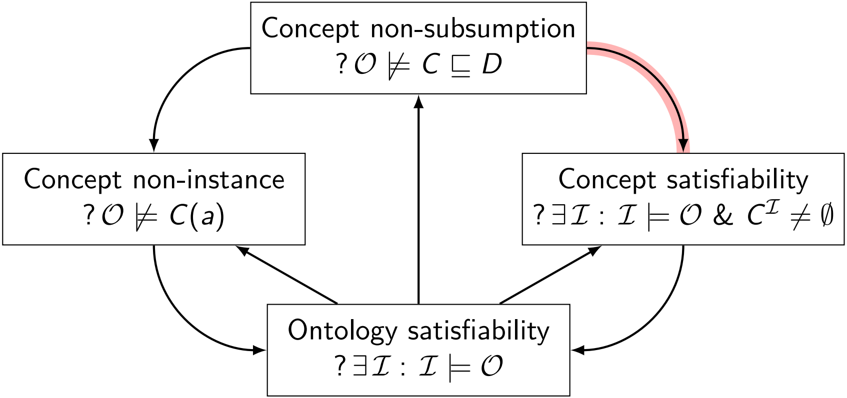
C is satisfiable w.r.t. $\mathcal{O}$ iff $\mathcal{O} \cup \{C(a)\}$ is satisfiable for every a not appearing in $\mathcal{O}$.

Proof.

- $(\Leftarrow)$: If $\mathcal{O} \cup \{C(a)\}$ is satisfiable then there exists a model $\mathcal{I} \models \mathcal{O} \cup \{C(a)\}$.

  Then $\mathcal{I} \models \mathcal{O}$ and $a^{\mathcal{I}} \in C^{\mathcal{I}} \neq \emptyset$. $\square$

Concept Non-Subsumption $\Rightarrow$ Concept Unsatisfiability



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$\mathcal{O} \not\models C \sqsubseteq D$ iff $C \sqcap \neg D$ is satisfiable w.r.t. $\mathcal{O}$.

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- $(\Rightarrow)$: If $\mathcal{O} \not\models C \sqsubseteq D$ then $\exists \mathcal{I} \models \mathcal{O}$: $\mathcal{I} \not\models C \sqsubseteq D$.
  Then $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$. Hence $\exists x \in (C^{\mathcal{I}} \setminus D^{\mathcal{I}}) = (C \sqcap \neg D)^{\mathcal{I}}$.

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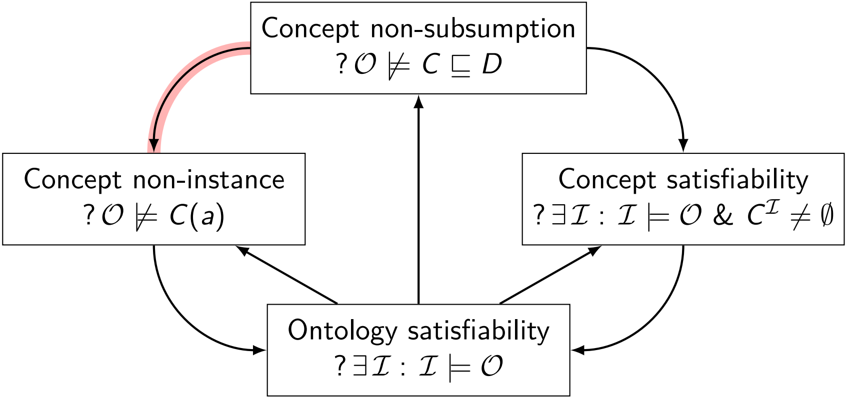
Concept Subsumption $\Rightarrow$ Concept Instance



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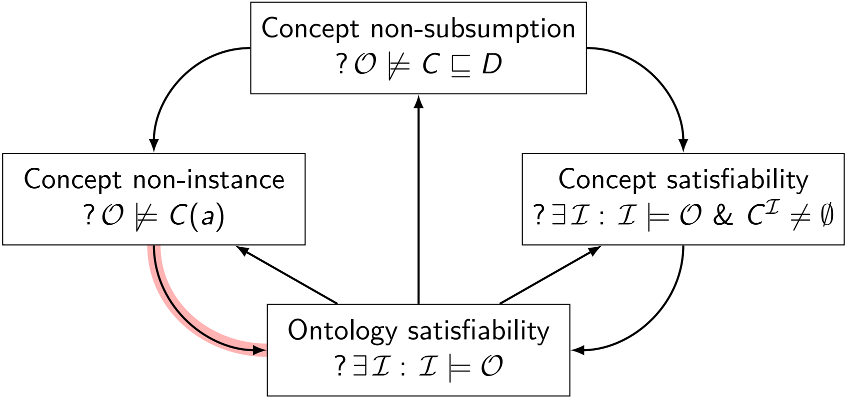
Concept Non-Instance ⇒ Ontology Satisfiability

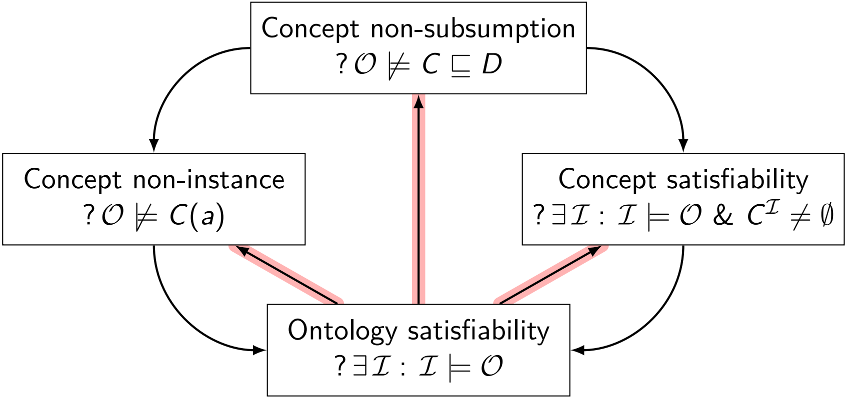


Concept Non-Instance $\Rightarrow$ Ontology Satisfiability

Lemma

$\mathcal{O} \not\models C(a)$ iff $\mathcal{O} \cup \{(\neg C)(a)\}$ is satisfiable.

Ontology Satisfiability $\Rightarrow$ Everything Else



Ontology Satisfiability $\Rightarrow$ Everything Else

Lemma

Then the following conditions are equivalent:

1. $\mathcal{O}$ *is unsatisfiable,*
2. $\top$ *is unsatisfiable w.r.t.* $\mathcal{O}$,
3. $\mathcal{O} \models \top \sqsubseteq \bot$,
4. $\mathcal{O} \models (\bot)(a)$ *for every* a,
5. $\mathcal{O} \models (\bot)(a)$ *for some* a.

Ontology Satisfiability $\Rightarrow$ Everything Else

Corollary

All standard reasoning problems are reducible to each other in polynomial time.

Outline

Description Logics

Tableau Procedures

- Deciding Concept Satisfiability
- Correctness of the Tableau Procedure
- Termination and Complexity Analysis
- Tableau with TBoxes
- Blocking

Axiom Pinpointing

Conclusions

Outline

Tableau

- We first focus first on pure concept satisfiability
  - Given: C a concept
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- A tableau is a directed labeled graph $T = (V, E, L)$ (most commonly, a tree) in which:
  - nodes V represent domain elements,
  - edges E represent pairs from role interpretations.
  - labeling function L assigns:

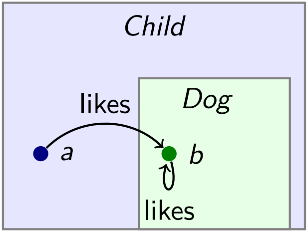
    to each $v \in V \quad \mapsto \quad$ a set of concepts $L(v)$

    to each $e \in E \quad \mapsto \quad$ a set of roles $L(e)$

Tableau: Example

Interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$:

- $\Delta^{\mathcal{I}} = \{a, b\}$
- $Child^{\mathcal{I}} = \{a, b\}$
- $Dog^{\mathcal{I}} = \{b\}$
- $likes^{\mathcal{I}} = \{(a, b), (b, b)\}$



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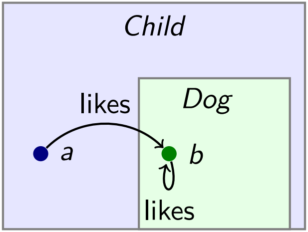
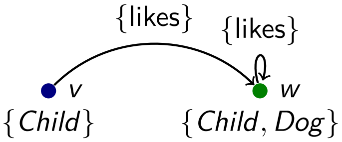


Tableau $T = (V, E, L)$:

- $V = \{v, w\}$
- $E = \{\langle v, w\rangle, \langle w, w\rangle\}$
- $L(v) = \{Child\}$
- $L(w) = \{Child, Dog\}$
- $L\langle v, w\rangle = L\langle w, w\rangle = \{likes\}$



Negation Normal Form

Before constructing a tableau for a concept C, it is first converted into a suitable normal form

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- Example: in NNF: $\forall R.(\neg A \sqcup \exists S.\neg B)$
  not in NNF: $\neg\exists R.A, \quad \forall R.\neg(A \sqcap B), \quad A \sqcap \exists R.\neg\top$

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  not in NNF: $\neg\exists R.A$, $\forall R.\neg(A \sqcap B)$, $A \sqcap \exists R.\neg\top$
- Transformation to NNF: "pushing negation inwards":
  - $\neg(C \sqcap D) \quad \Rightarrow \quad (\neg C) \sqcup (\neg D)$
  - $\neg(C \sqcup D) \quad \Rightarrow \quad (\neg C) \sqcap (\neg D)$
  - $\neg(\exists R.C) \quad \Rightarrow \quad \forall R.(\neg C)$
  - $\neg(\forall R.C) \quad \Rightarrow \quad \exists R.(\neg C)$
  - $\neg\neg C \quad \Rightarrow \quad C$
  - $\neg\top \quad \Rightarrow \quad \bot$, $\neg\bot \quad \Rightarrow \quad \top$

Tableau Expansion Rules

- To check satisfiability of a concept C in NNF, create a node x and set $L(x) = \{C\}$. We call it tableau initialization rule.

$x \bullet C$

Example: $C = \exists R.A \sqcap (\forall R.(\neg A) \sqcup B)$

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- $\sqcap$-Rule:
  if $(A \sqcap B) \in L(x)$ and $\{A, B\} \not\subseteq L(x)$
  then update $L(x) := L(x) \cup \{A, B\}$

$x \bullet A \sqcap B \quad \rightsquigarrow \quad x \bullet A \sqcap B, A, B$

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- $\sqcup$-Rule:
  if $(A \sqcup B) \in L(x)$ and $\{A, B\} \cap L(x) = \emptyset$
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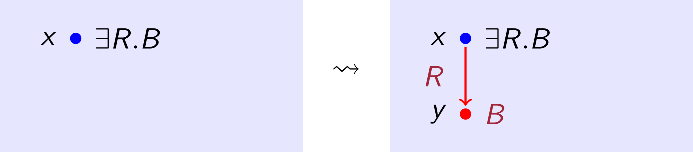
Tableau Expansion Rules

- $\exists$-Rule:
  if $(\exists R.B) \in L(x)$ and $B \notin L(y)$ for all y with $R \in L\langle x, y\rangle$
  then create a new y and set $L\langle x, y\rangle := \{R\}$ and $L(y) := \{B\}$



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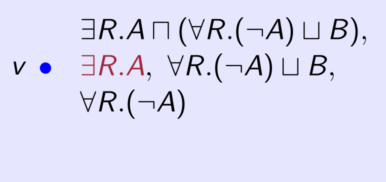

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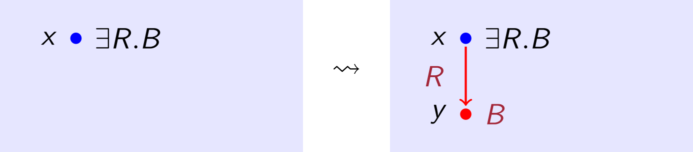

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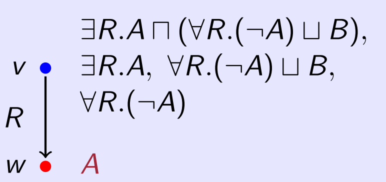

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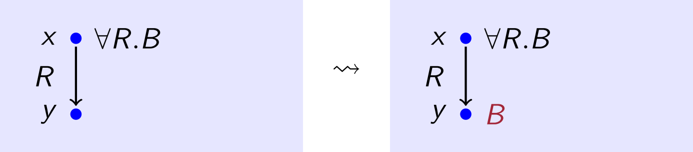

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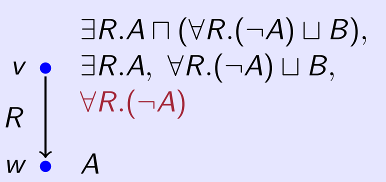

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$$\begin{tikzcd} x \arrow[d, "R"'] & \forall R.B \\ y & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} x \arrow[d, "R"'] & \forall R.B \\ y & B \end{tikzcd}$$

Example: $C = \exists R.A \sqcap (\forall R.(\neg A) \sqcup B)$

$$\begin{tikzcd} v \arrow[d, "R"'] & \exists R.A \sqcap (\forall R.(\neg A) \sqcup B),\ \exists R.A,\ \forall R.(\neg A) \sqcup B,\ \forall R.(\neg A) \\ w & A,\ \neg A \end{tikzcd}$$

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- $\bot$-Rule:
  if $\{A, \neg A\} \subseteq L(x)$ and $\bot \notin L(x)$
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$x \bullet A, \neg A \quad \rightsquigarrow \quad x \bullet A, \neg A, \bot$

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$$
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 & & \exists R.A,\ \forall R.(\neg A) \sqcup B, \\
 & & \forall R.(\neg A) \\
R & \downarrow & \\
w & \bullet & A,\ \neg A
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Tableau Expansion Rules

Example: $C = \exists R.A \sqcap (\forall R.(\neg A) \sqcup B)$ (two tableau expansions)

v: $\exists R.A \sqcap (\forall R.(\neg A) \sqcup B)$, $\exists R.A$, $\forall R.(\neg A) \sqcup B$, $\forall R.(\neg A)$

$v \xrightarrow{R} w$

w: A, $\neg A$, $\perp$

v: $\exists R.A \sqcap (\forall R.(\neg A) \sqcup B)$, $\exists R.A$, $\forall R.(\neg A) \sqcup B$, B

$v \xrightarrow{R} w$

w: A

Outline

Completeness of Tableau

Definition

A tableau $T = (V, E, L)$ contains a clash if $\perp \in L(x)$ for some node $x \in V$. A tableau is clash-free if it does not contain a clash.

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Theorem (Completeness)

If an $\mathcal{ALC}$ concept C is satisfiable then the tableau rules can be always applied in such a way that a clash is never produced.

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Proof Idea.

Use a model of C to guide construction of the tableau. □

Completeness: Proof Idea

Example: $C = \exists R.A \sqcap (\forall R.(\neg A) \sqcup B)$

Model $\mathcal{I}$:

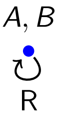
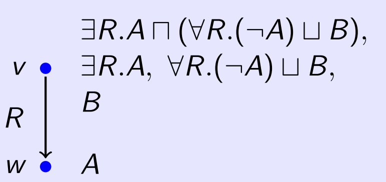


Tableau $T = (V, E, L)$:



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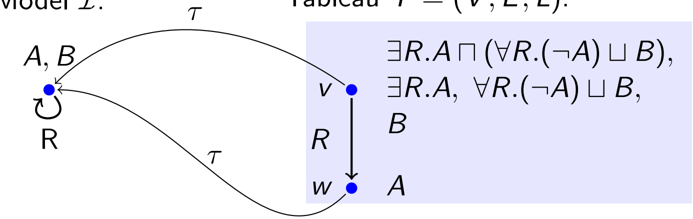

- For every created node x we assign the corresponding element $\tau(x) \in \Delta^{\mathcal{I}}$ of the model $\mathcal{I}$.

Completeness: Proof Idea

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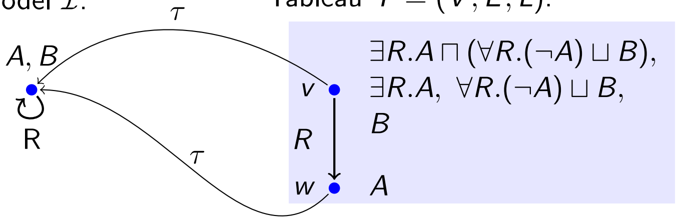

- For every created node x we assign the corresponding element $\tau(x) \in \Delta^{\mathcal{I}}$ of the model $\mathcal{I}$.
- We can always apply the rules such that:
  1. if $D \in L(x)$ then $\tau(x) \in D^{\mathcal{I}}$, and
  2. if $R \in L(x, y)$ then $\langle \tau(x), \tau(y) \rangle \in R^{\mathcal{I}}$.

Soundness of Tableau

Theorem (Soundness)

If there exists a clash-free tableau $T = (V, E, L)$ *such that* $C \in L(v)$ *for some* $v \in V$*, then* C *is satisfiable.*

Soundness of Tableau

Theorem (Soundness)

If there exists a clash-free fully expanded tableau $T = (V, E, L)$ such that $C \in L(v)$ for some $v \in V$, then C is satisfiable.

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A tableau is fully expanded if all expansion rules are applied to every node.

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Proof Idea.

Build a model from the tableau. □

Soundness: Proof Idea

Tableau $T = (V, E, L)$:

$$\begin{tikzcd} v \arrow[d, "R"'] & \exists R.A \sqcap (\forall R.(\neg A) \sqcup B), \\ & \exists R.A,\ \forall R.(\neg A) \sqcup B, \\ & B \\ w & A \end{tikzcd}$$

Soundness: Proof Idea

Tableau $T = (V, E, L)$:

v: $\exists R.A \sqcap (\forall R.(\neg A) \sqcup B)$, $\exists R.A$, $\forall R.(\neg A) \sqcup B$, B

$$v \xrightarrow{R} w$$

w: A

Model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$:

- $\Delta^{\mathcal{I}} = \{v, w\}$
- $A^{\mathcal{I}} = \{w\}$
- $B^{\mathcal{I}} = \{v\}$
- $R^{\mathcal{I}} = \{\langle v, w\rangle\}$

Soundness: Proof Idea

Tableau $T = (V, E, L)$:

$$
\begin{array}{ll}
v & L(v) = \{\exists R.A \sqcap (\forall R.(\neg A) \sqcup B),\ \exists R.A,\ \forall R.(\neg A) \sqcup B,\ B\} \\
\downarrow R & \\
w & L(w) = \{A\}
\end{array}
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Model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$:

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- The model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is defined from $T = (V, E, L)$ by:
  - $\Delta^{\mathcal{I}} = V$
  - $A^{\mathcal{I}} = \{x \mid A \in L(x)\}$
  - $R^{\mathcal{I}} = \{\langle x, y\rangle \mid R \in L(x, y)\}$

Soundness: Proof Idea

Tableau $T = (V, E, L)$:

v: $\exists R.A \sqcap (\forall R.(\neg A) \sqcup B)$, $\exists R.A$, $\forall R.(\neg A) \sqcup B$, B

$v \xrightarrow{R} w$

w: A

Model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$:

- $\Delta^{\mathcal{I}} = \{v, w\}$
- $A^{\mathcal{I}} = \{w\}$
- $B^{\mathcal{I}} = \{v\}$
- $R^{\mathcal{I}} = \{\langle v, w\rangle\}$

- The model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is defined from $T = (V, E, L)$ by:
  - $\Delta^{\mathcal{I}} = V$
  - $A^{\mathcal{I}} = \{x \mid A \in L(x)\}$
  - $R^{\mathcal{I}} = \{\langle x, y\rangle \mid R \in L(x, y)\}$
- By structural induction it can be shown that:

$$\boxed{\text{if } D \in L(x) \text{ then } x \in D^{\mathcal{I}}}$$

Outline

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Is outcome 3 possible?

Properties of Tableau Expansion Rules

- $x \bullet C$
- $x \bullet A \sqcap B \rightsquigarrow A, B$ $\quad$ $x \bullet A \sqcup B \rightsquigarrow A \mid B$
- $$\begin{tikzcd} x \bullet \exists R.B \arrow[d, "\rightsquigarrow R"'] \\ y \bullet \rightsquigarrow B \end{tikzcd} \qquad \begin{tikzcd} x \bullet \forall R.B \arrow[d, "R"'] \\ y \bullet \rightsquigarrow B \end{tikzcd}$$
- $x \bullet A, \neg A \rightsquigarrow \perp$

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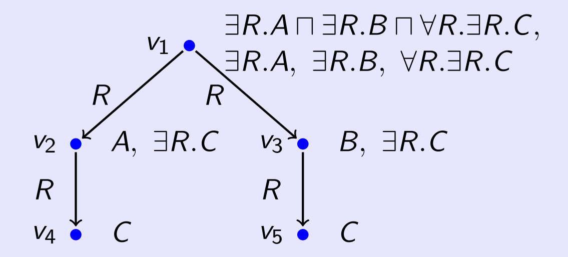
2. There can be at most one predecessor of every node

Properties of Tableau



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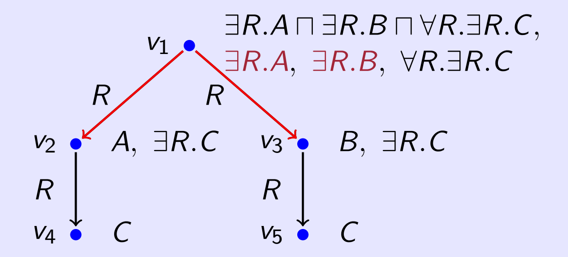

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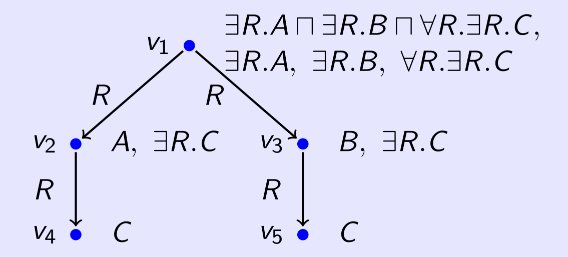

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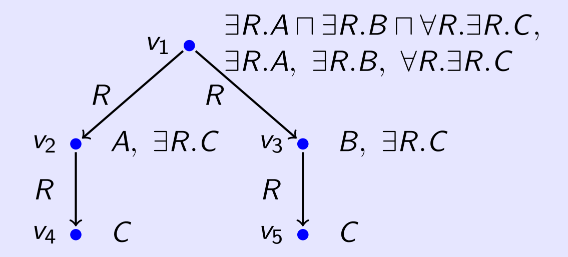

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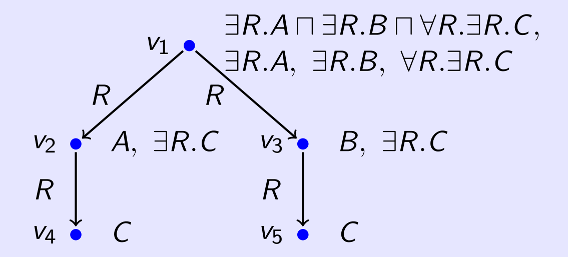

tableau contains $\leq \|C\| \times \|C\|^{qd(C)}$ concepts overall

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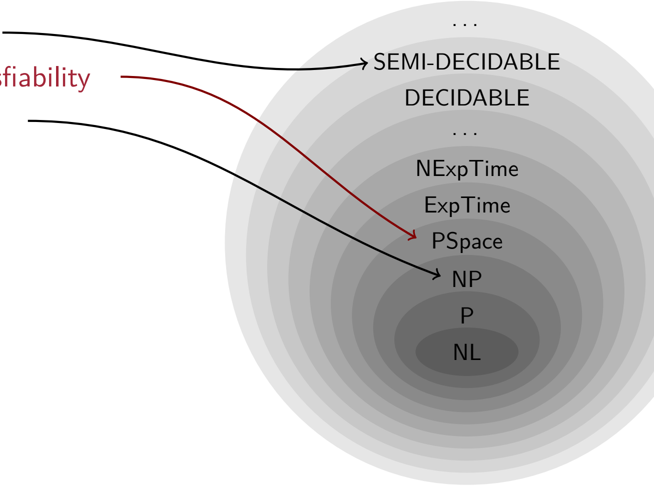
- In fact, it is possible to decide concept satisfiability in polynomial space (PSpace $\subseteq$ ExpTime $\subseteq$ NExpTime)

Complexity of the Tableau Procedure

- First-Order Logic
- $\mathcal{ALC}$ Concept Satisfiability
- Propositional Logic



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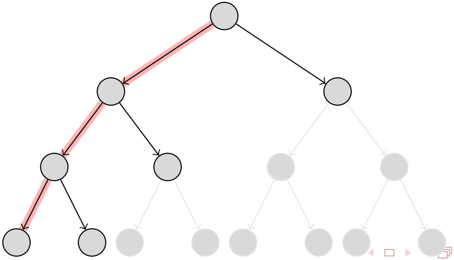

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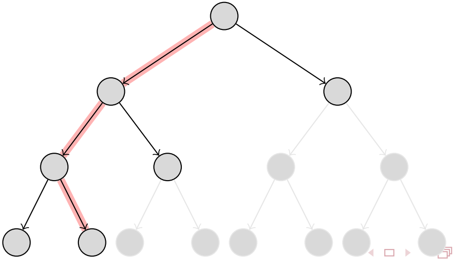

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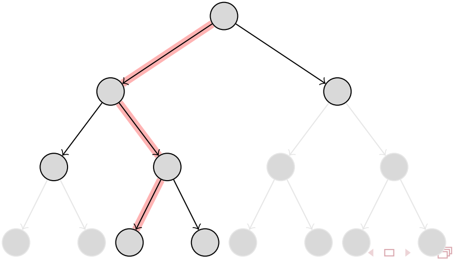

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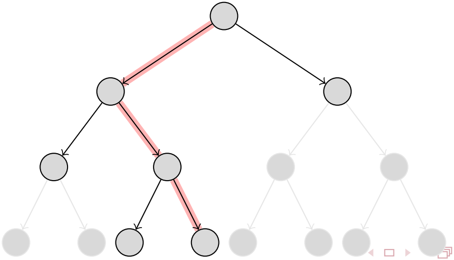

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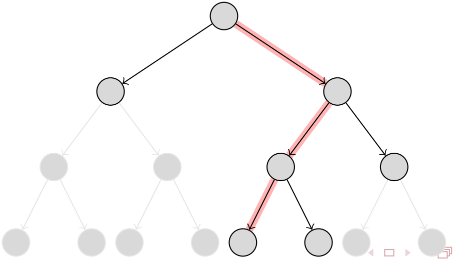

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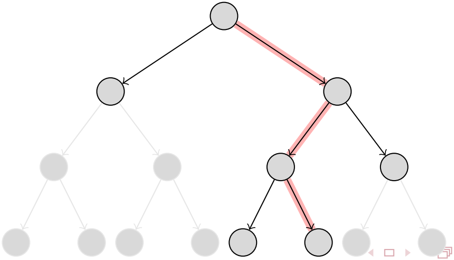

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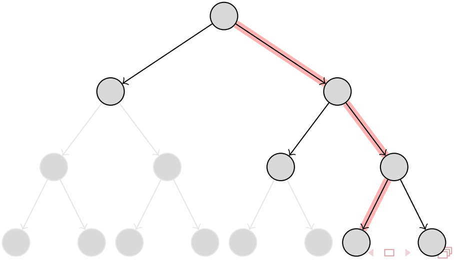

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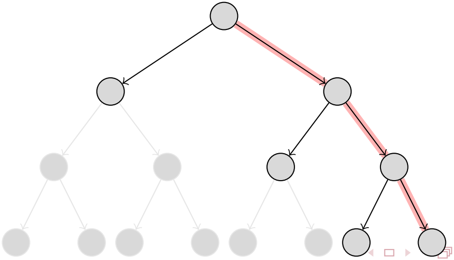

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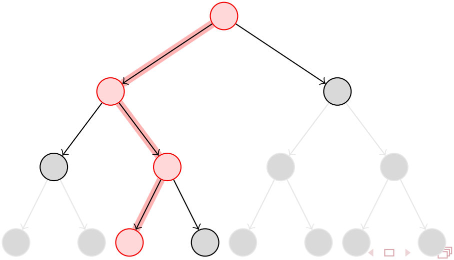
- Expand the tableau depth-first and keep only one branch in memory.
- Once all nodes on a branch are fully expanded, nothing new can be added to these nodes anymore.



Outline

Concept Satisfiability w.r.t. TBox Axioms

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- Our goal is to extend the tableau procedure so that we can test satisfiability of a concept C w.r.t. an ontology $\mathcal{O}$
- Recall that C is satisfiable w.r.t. $\mathcal{O}$ if there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$ and $\mathcal{I} \models \mathcal{O}$
- For simplicity, assume that $\mathcal{O}$ contains only of TBox axioms:
  - concept inclusions $C \sqsubseteq D$
  - concept equivalences $C \equiv D$

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  3. $\top \sqsubseteq C \quad \rightsquigarrow \quad \top \sqsubseteq NNF(C)$
- Note that $\mathcal{I} \models \top \sqsubseteq C$ if and only if $C^{\mathcal{I}} = \Delta^{\mathcal{I}}$.
  - or $d \in C^{\mathcal{I}}$ for every $d \in \Delta^{\mathcal{I}}$

Tableau Rules for TBox Axioms

To take TBox axioms into account, we need to add one more rule:

$\sqcap$-Rule: if $(A \sqcap B) \in L(x)$ and $\{A, B\} \not\subseteq L(x)$
then update $L(x) := L(x) \cup \{A, B\}$

$\sqcup$-Rule: if $(A \sqcup B) \in L(x)$ and $\{A, B\} \cap L(x) = \emptyset$
then update $L(x) := L(x) \cup \{A\}$ or $L(x) := L(x) \cup \{B\}$

$\exists$-Rule: if $(\exists R.B) \in L(x)$ and $B \notin L(y)$ for all y with $R \in L(x, y)$
then create a new y and set $L(x, y) := \{R\}$ and $L(y) := \{B\}$

$\forall$-Rule: if $(\forall R.B) \in L(x)$ and $R \in L(x, y)$, $B \notin L(y)$ for some $y \in V$
then update $L(y) := L(y) \cup \{B\}$

$\bot$-Rule: if $\{A, \neg A\} \subseteq L(x)$ and $\bot \notin L(x)$
then update $L(x) := L(x) \cup \{\bot\}$

$\boxed{\top\text{-Rule:}}$ if $\top \sqsubseteq C \in \mathcal{O}$ and $C \notin L(x)$
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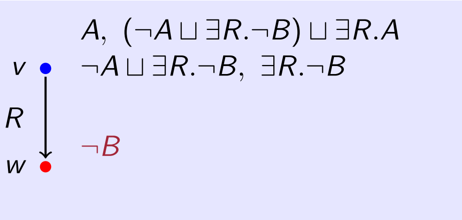

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  $\rightsquigarrow \quad \top \sqsubseteq \neg(A \sqcap \forall R.B) \sqcup \exists R.A$

  $\rightsquigarrow \quad \top \sqsubseteq (\neg A \sqcup \exists R.\neg B) \sqcup \exists R.A$
- Let's check satisfiability of A w.r.t. $\mathcal{O}$:
  - $\top$-Rule: $L(w) := L(w) \cup \{(\neg A \sqcup \exists R.\neg B) \sqcup \exists R.A\}$

$$\begin{array}{ll} & A,\ (\neg A \sqcup \exists R.\neg B) \sqcup \exists R.A \\ v & \neg A \sqcup \exists R.\neg B,\ \exists R.\neg B \\ \downarrow R & \\ w & \neg B \end{array}$$

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$v \xrightarrow{R} w$

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Completeness of Tableau for TBoxes

Theorem (Completeness)

If an $\mathcal{ALC}$ concept C is satisfiable w.r.t. $\mathcal{O}$ then the tableau rules can be always applied in such a way that a clash is never produced.

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Proof.

As before, we build the tableau $T = (V, E, L)$ by applying the rules to mimic $\mathcal{I} \models \mathcal{O}$ such that $C^{\mathcal{I}} \neq \emptyset$. $\square$

Soundness of Tableau for TBoxes

Theorem (Soundness)

If there exists a clash-free fully expanded tableau $T = (V, E, L)$ *such that* $C \in L(v)$ *for some* $v \in V$*, then* C *is satisfiable w.r.t.* $\mathcal{O}$.

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Proof.

As in the case without $\mathcal{O}$, we define an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ from $T = (V, E, L)$ with $\Delta^{\mathcal{I}} = V$ and prove that:

if $D \in L(x)$ then $x \in D^{\mathcal{I}}$

This implies that $C^{\mathcal{I}} \neq \emptyset$.

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This implies that $C^\mathcal{I} \neq \emptyset$. Now we also prove that $\mathcal{I} \models \mathcal{O}$:

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This implies that $C^{\mathcal{I}} \neq \emptyset$. Now we also prove that $\mathcal{I} \models \mathcal{O}$:

Take any $\top \sqsubseteq D \in \mathcal{O}$ and $x \in \top^{\mathcal{I}} = \Delta^{\mathcal{I}} = V$.

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Take any $\top \sqsubseteq D \in \mathcal{O}$ and $x \in \top^{\mathcal{I}} = \Delta^{\mathcal{I}} = V$.
Then $D \in L(x)$ because $\top$-Rule is applied to x.
Then $x \in D^{\mathcal{I}}$. Hence $\mathcal{I} \models \top \sqsubseteq D$. $\square$

Termination

Three possible outcomes of tableau rules application:

1. Tableau can be fully expanded without producing clash.
   $\Rightarrow$ In this case the concept is satisfiable.
2. Every attempt to apply the rules eventually results in a clash.
   $\Rightarrow$ In this case the concept is unsatisfiable.
3. The rules can be applied forever without producing a clash.
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Unfortunately, with TBoxes, the outcome 3 becomes possible!

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v_0 • $A,\ \neg A \sqcup \exists R.A$

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$$\begin{tikzcd} v_0 \bullet \arrow[d, "R"'] & A,\ \neg A \sqcup \exists R.A,\ \exists R.A \\ v_1 \bullet & A \end{tikzcd}$$

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v_2 A

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- Let's check satisfiability of A w.r.t. $\mathcal{O}$:
- This process can continue forever!

$$
\begin{tikzcd}
v_0 \arrow[d, "R"'] & A,\ \neg A \sqcup \exists R.A,\ \exists R.A \\
v_1 \arrow[d, "R"'] & A,\ \neg A \sqcup \exists R.A,\ \exists R.A \\
v_2 \arrow[d, dotted] & A,\ \neg A \sqcup \exists R.A,\ \exists R.A \\
{} &
\end{tikzcd}
$$

Outline

Blocking

- Notice that the labels of the nodes repeat:

$$
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Definition (Blocking)

A node $v \in V$ is blocked if there exists an ancestor node $w \in V$ of v such that $L(v) \subseteq L(w)$. (We say that v is blocked by w.)

Blocking

- Notice that the labels of the nodes repeat:

$$\begin{array}{lll} v_0 & \bullet & A,\ \neg A \sqcup \exists R.A,\ \exists R.A \\ R & \downarrow & \\ v_1 & \bullet & A,\ \neg A \sqcup \exists R.A,\ \exists R.A \\ & \vdots & \end{array}$$

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- Above v_1 is blocked by v_0, but v_0 is not blocked by v_1

Soundness with Blocking

Definition

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Theorem (Soundness)

If there exists a clash-free fully expanded tableau $T = (V, E, L)$ such that $C \in L(v)$ for some $v \in V$, then C is satisfiable w.r.t. $\mathcal{O}$.

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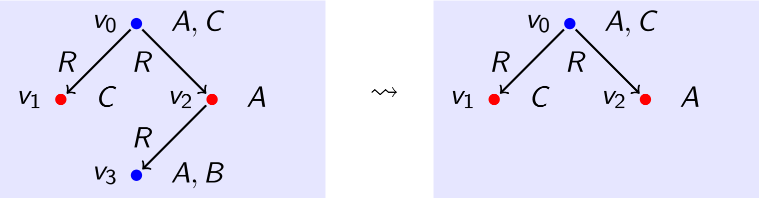

  - v_3 is not blocked, but it is a descendant of a blocked node v_2
  - no rules need to be applied as v_3 can be simply removed

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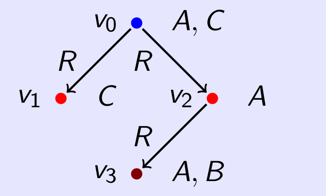

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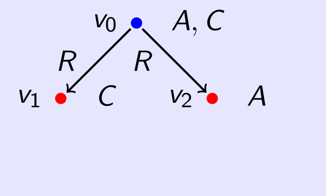

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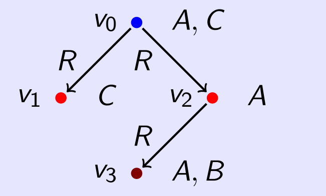

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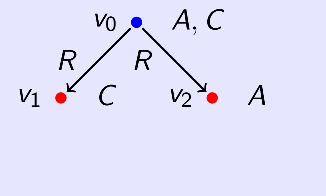

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- However, the optimal complexity for $\mathcal{ALC}$ concept satisfiability w.r.t. TBoxes, is "only" ExpTime.

Complexity

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- $\mathcal{ALC}$ Concept Satisfiability w.r.t. TBox
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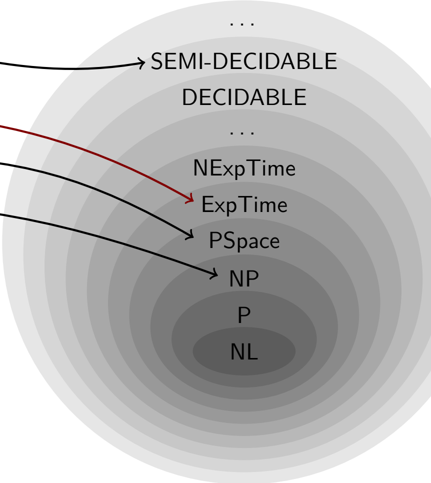

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- Justifications
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- The axiom pinpointing algorithms can be used to narrow down the axioms responsible for the error
  - using a series of entailment tests

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- There are 2^n justifications:
  - for every i choose either $A_{i-1} \sqsubseteq B \sqcap A_i$ or $A_{i-1} \sqsubseteq C \sqcap A_i$

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- Next we describe a more goal-directed algorithm

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- The Hitting Set Tree algorithm (short: HST-algorithm) explores such tuples $\langle\beta_1, \ldots, \beta_n\rangle$ in a systematic way

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The Hitting Set Tree (short: HS-tree) for $\mathcal{O} \models \alpha$ is a labeled tree such that:

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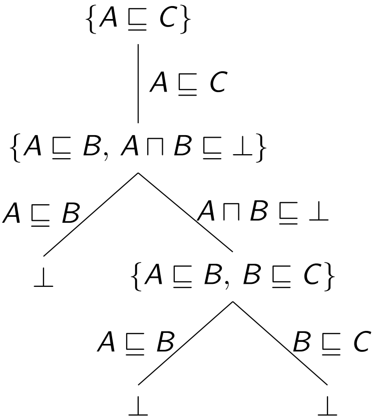

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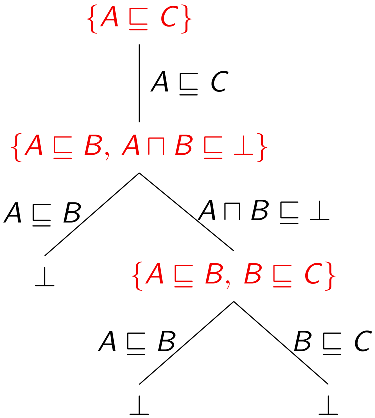

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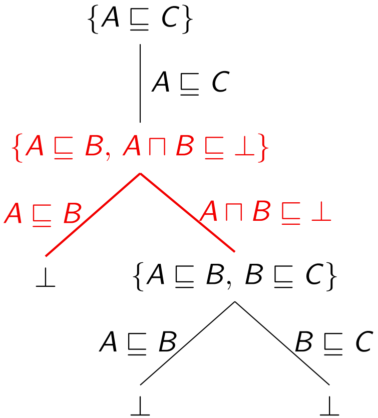

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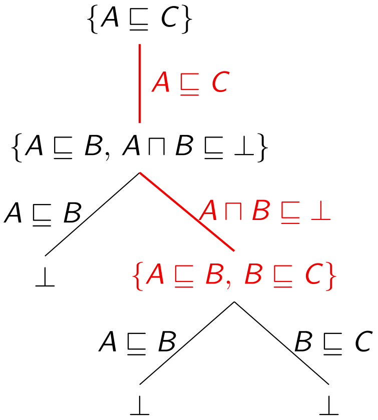

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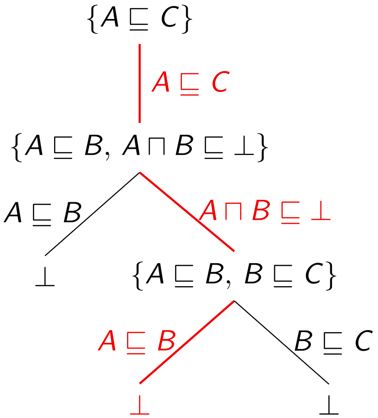

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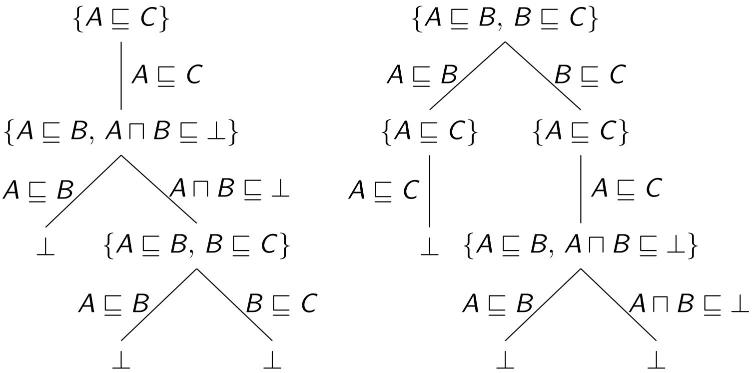

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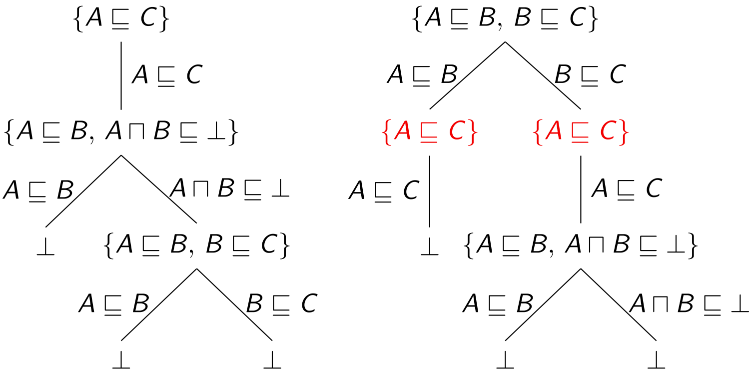

Note that a HS-tree may contain a justification multiple times!

Properties of Hitting Set Trees

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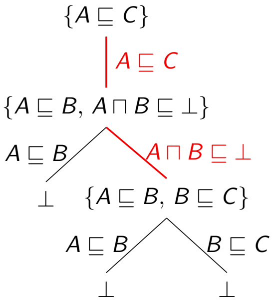

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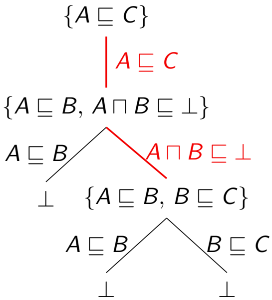

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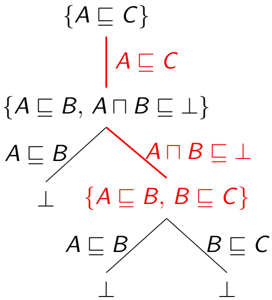

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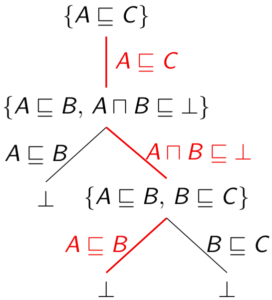

3. Each HS-tree contains at most exponentially-many nodes
   - every path is labeled by a unique sequence of different axioms

Construction of a HS-Tree

A HS-Tree for $\mathcal{O} \models \alpha$ can be constructed as follows:

1. Create a root node v_0
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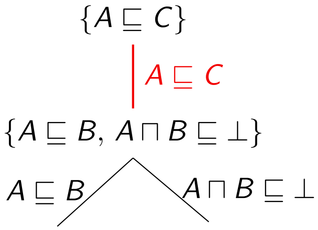

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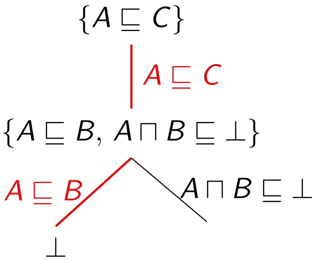

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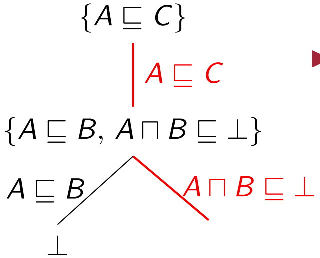

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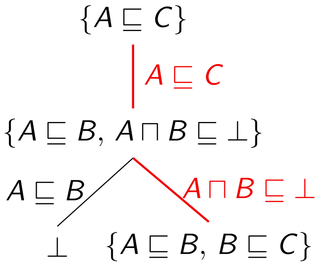

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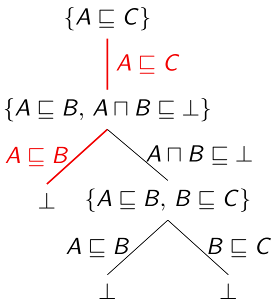

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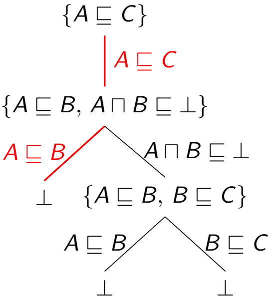

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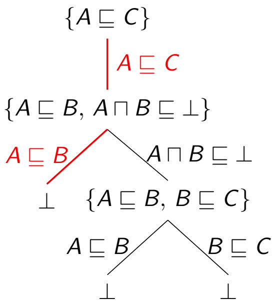

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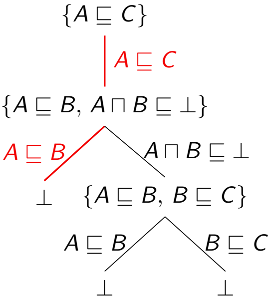

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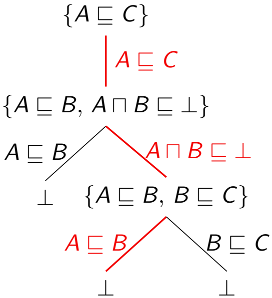

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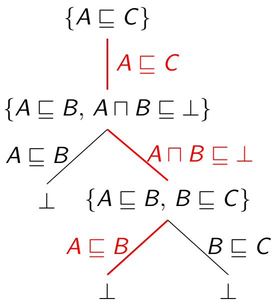

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$\Rightarrow$ Flexible trade-off: memory vs speed

Outline

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$\Rightarrow$ Each (minimal) repair is a (minimal) hitting set of all justifications

- Gives the name of the HST-algorithm

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- Question: how to find M satisfying 1 and 2?

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- For each axiom $\beta \in \mathcal{O}$ introduce a propositional variable p_β
- Goal: find a model $\mathcal{I}$ such that $p_\beta^\mathcal{I} = 1$ iff $\beta \in M$
- Then the conditions can be expressed by the formula:

$$F = \bigwedge_{i=1}^{n} \bigvee_{\beta \in J_i} \neg p_\beta \quad \wedge \quad \bigwedge_{j=1}^{m} \bigvee_{\beta \in R_j} p_\beta$$

1. M must miss some $\beta \in J_i$ for each i $(1 \leq i \leq n)$
2. M must contain some $\beta \in R_j$ for each j $(1 \leq j \leq n)$

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- $\mathcal{O} \setminus M = \{B \sqsubseteq C, A \sqsubseteq C, A \sqcap B \sqsubseteq \bot\}$ is a new repair
  - in fact, even a new minimal repair

The SAT-Based Algorithm

1. Set $F = \top$
2. While F is satisfiable do:
   - take any model $\mathcal{I}$ of F
   - define $M = \{\beta \mid p_\beta^\mathcal{I} = 1\}$
   - if $M \models \alpha$ then extract a justification $J \subseteq M$
   - otherwise set $R = \mathcal{O} \setminus M$ (and optionally minimize)
   - update F based on J or R
3. Return all computed justifications J (and / or repairs R)

Comparison of Axiom Pinpointing Methods

	HST	SAT
Repetition of justifications:	may repeat(*)	no repetition
Memory consumption:	polynomial	exponential

(*)There is an example in which justifications repeat exponentially-many times

Outline

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Tableau Procedures

Axiom Pinpointing

Conclusions

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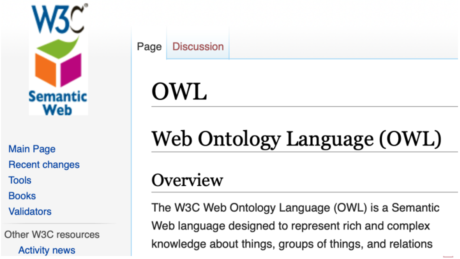

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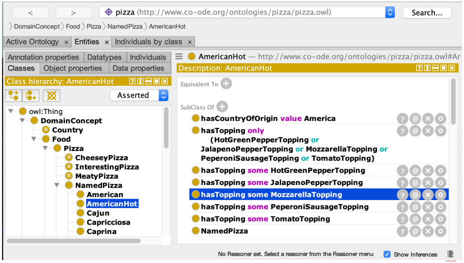

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FaCT++	CB
HermiT	ConDOR
Konclude	ELK
MoRE	Sequoia
Pellet	
RACER	
TrOWL	

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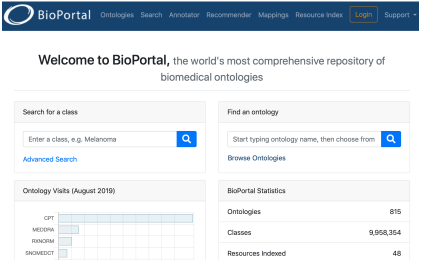

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  - work by deriving consequences, not building models

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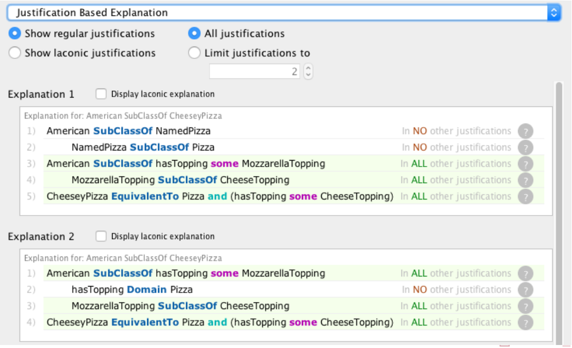

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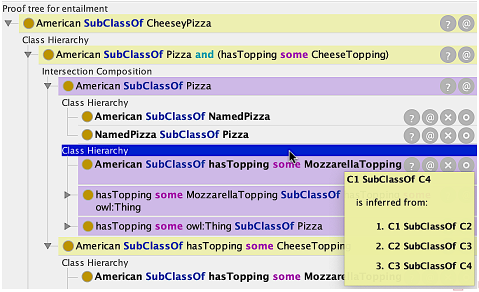
  - other applications, e.g., inconsistency-tolerant reasoning
- Implemented in many tools
  - explanation workbench, EL+SAT, EL2MUS, SATPin,...
- Other explanation methods: proof-based explanations
  - show how consequences are derived from axioms



Algorithms

Lessons learned about algorithms in general:

1. Never neglect correctness!
2. Worst-case complexity may be misleading
3. Goal-directed behavior is the key to practical efficiency
4. Only empirical evaluation can give a complete picture